Interval total colorings of graphs

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A total coloring of a graph G is a coloring of its vertices and edges such that no adjacent vertices, edges, and no incident vertices and edges obtain the same color. An *interval total t-coloring* of a graph G is a total coloring of G with colors $1, 2, \ldots, t$ such that at least one vertex or edge of G is colored by i, $i = 1, 2, \ldots, t$, and the edges incident to each vertex v together with v are colored by $d_G(v) + 1$ consecutive colors, where $d_G(v)$ is the degree of the vertex v in G. In this paper we investigate some properties of interval total colorings. We also determine exact values of the least and the greatest possible number of colors in such colorings for some classes of graphs.

Keywords: total coloring, interval coloring, connected graph, regular graph, bipartite graph

1. Introduction

A total coloring of a graph G is a coloring of its vertices and edges such that no adjacent vertices, edges, and no incident vertices and edges obtain the same color. The concept of total coloring was introduced by V. Vizing [22] and independently by M. Behzad [4]. The total chromatic number $\chi''(G)$ is the smallest number of colors needed for total coloring of G. In 1965 V. Vizing and M. Behzad conjectured that $\chi''(G) \leq \Delta(G) + 2$ for every graph G [4, 22], where $\Delta(G)$ is the maximum degree of a vertex in G. This conjecture became known as Total Coloring Conjecture [10]. It is known that Total Coloring Conjecture holds for cycles, for complete graphs [5], for bipartite graphs, for complete multipartite graphs [25], for graphs with a small maximum degree [11, 12, 18, 21], for graphs with minimum degree at least $\frac{3}{4}|V(G)|$ [9], and for planar graphs G with $\Delta(G) \neq 6$ [6, 10, 20]. M. Rosenfeld [18] and N. Vijayaditya [21] independently proved that the total chromatic number of graphs G with $\Delta(G) = 3$ is at most 5. A. Kostochka in [11] proved that the total chromatic number of graphs with $\Delta(G) = 4$ is at most 6. Later, also he in [

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12] proved that the total chromatic number of graphs with $\Delta(G) = 5$ is at most 7. The general upper bound for the total chromatic number was obtained by M. Molloy and B. Reed [15], who proved that $\chi''(G) \leq \Delta(G) + 10^{26}$ for every graph G. The exact value of the total chromatic number is known only for paths, cycles, complete and complete bipartite graphs [5], n-dimensional cubes, complete multipartite graphs of odd order [8], outerplanar graphs [26] and planar graphs G with $\Delta(G) \geq 9$ [7, 10, 13, 23].

The key concept discussed in a present paper is the following. Given a graph G, we say that G is interval total colorable if there is $t \geq 1$ for which G has a total coloring with colors $1, 2, \ldots, t$ such that at least one vertex or edge of G is colored by i, $1, 2, \ldots, t$, and the edges incident to each vertex v together with v are colored by $d_G(v) + 1$ consecutive colors, where $d_G(v)$ is the degree of the vertex v in G.

The concept of interval total coloring [16, 17] is a new one in graph coloring, synthesizing interval colorings [1, 2] and total colorings. The introduced concept is valuable as it connects to the problems of constructing a timetable without a "gap" and it extends to total colorings of graphs one of the most important notions of classical mathematics - the one of continuity.

In this paper we investigate some properties of interval total colorings of graphs. Also, we show that simple cycles, complete graphs, wheels, trees, regular bipartite graphs and complete bipartite graphs have interval total colorings. Moreover, we obtain some bounds for the least and the greatest possible number of colors in interval total colorings of these graphs.

2. Definitions and preliminary results

All graphs considered in this work are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. An (a, b)-biregular bipartite graph G is a bipartite graph G with the vertices in one part having degree a and the vertices in the other part having degree b. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of vertices in G by $\Delta(G)$, the diameter of G by $diam(G)$, the chromatic number of G by $\chi(G)$ and the edge-chromatic number of G by $\chi'(G)$. A vertex u of a graph G is universal if $d_G(u) = |V(G)| - 1$. A *proper edge-coloring* of a graph G is a coloring of the edges of G such that no two adjacent edges receive the same color. For a total coloring α of a graph G and for any $v \in V(G)$, define the set $S[v, \alpha]$ as follows:

$$S[v, \alpha] = \{\alpha(v)\} \cup \{\alpha(e) \mid e \text{ is incident to } v\}$$

Let $\lfloor a \rfloor$ ($\lceil a \rceil$) denote the greatest (the least) integer $\leq a$ ($\geq a$). For two integers $a \leq b$, the set $\{a, a+1, \ldots, b\}$ is denoted by $[a, b]$.

An *interval t-coloring* of a graph G is a proper edge-coloring of G with colors $1, 2, \ldots, t$ such that at least one edge of G is colored by i, $i = 1, 2, \ldots, t$, and the edges incident to each vertex v are colored by $d_G(v)$ consecutive colors. A graph G is interval colorable if there is $t \geq 1$ for which G has an interval t-coloring. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the greatest value of t for which G has an interval t-coloring is denoted by $W(G)$.

An *interval total t-coloring* of a graph G is a total coloring of G with colors $1, 2, \ldots, t$ such that at least one vertex or edge of G is colored by i, $i = 1, 2, \ldots, t$, and the edges incident to each vertex v together with v are colored by $d_G(v) + 1$ consecutive colors.

For $t \geq 1$, let $\mathfrak{T}_t$ denote the set of graphs which have an interval total t-coloring, and assume: $\mathfrak{T} = \bigcup_{t \geq 1} \mathfrak{T}_t$. For a graph $G \in \mathfrak{T}$, the least and the greatest values of t for which $G \in \mathfrak{T}_t$ are denoted by $w_\tau(G)$ and $W_\tau(G)$, respectively. Clearly,

$$\chi''(G) \leq w_\tau(G) \leq W_\tau(G) \leq |V(G)| + |E(G)| \text{ for every graph } G \in \mathfrak{T}.$$

Terms and concepts that we do not define can be found in [24, 25].

We will use the following two results.

Theorem 1 *[1, 2]. If G is a connected triangle-free graph and $G \in \mathfrak{N}$, then*

$$W(G) \leq |V(G)| - 1.$$

Theorem 2 *[3]. If G is a connected (a, b)-biregular bipartite graph with $|V(G)| \geq 2(a+b)$ and $G \in \mathfrak{N}$, then*

$$W(G) \leq |V(G)| - 3.$$

3. Some properties of interval total colorings of graphs

First we prove a simple property of interval total colorings that for any interval total coloring of a graph G there is an inverse interval total coloring of the same graph.

Proposition 3 *If α is an interval total t-coloring of a graph G, then a total coloring β, where*
1) $\beta(v) = t + 1 - \alpha(v)$ for each $v \in V(G)$,
2) $\beta(e) = t + 1 - \alpha(e)$ for each $e \in E(G)$,
is also an interval total t-coloring of a graph G.

Proof. Clearly, a total coloring β contains at least one vertex or edge with color i, $i = 1, 2, \ldots, t$. Since $S[v, \alpha]$ is an interval for each $v \in V(G)$, hence $S[v, \alpha] = [a, b]$. By the definition of the coloring β it follows that $S[v, \beta] = [t + 1 - b, t + 1 - a]$ for each $v \in V(G)$. $\square$

Next we prove the proposition which implies that in definition of interval total t-coloring, the requirement that every color i, $i = 1, 2, \ldots, t$, appear in an interval total t-coloring isn't necessary in the case of connected graphs.

Proposition 4 *Let α be a total coloring of the connected graph G with colors $1, 2, \ldots, t$ such that the edges incident to each vertex $v \in V(G)$ together with v are colored by $d_G(v)+1$ consecutive colors, and $\min_{v \in V(G), e \in E(G)} \{\alpha(v), \alpha(e)\} = 1$, $\max_{v \in V(G), e \in E(G)} \{\alpha(v), \alpha(e)\} = t$. Then α is an interval total t-coloring of G.*

Proof. For the proof of the proposition it suffices to show that if $t \geq 3$, then for color s, $1 < s < t$, there exists at least one vertex or edge of G which is colored by s. We consider four possible cases.

Case 1: there are vertices $v, v' \in V(G)$ such that $\alpha(v) = 1$, $\alpha(v') = W_\tau(G)$.

Since G is connected, there exists a simple path P_1 joining v with v', where

$$P_1 = (v_0, e_1, v_1, \ldots, v_{i-1}, e_i, v_i, \ldots, v_{k-1}, e_k, v_k), \ v_0 = v, \ v_k = v'.$$

If $\alpha(v_i) \neq s$, $i = 1, 2, \ldots, k-1$, and $\alpha(e_j) \neq s$, $j = 1, 2, \ldots, k$, then there exists an index i_0, $1 \leq i_0 < k$, such that $\alpha(e_{i_0}) < s$ and $\alpha(e_{i_0+1}) > s$. Hence, there is an edge of G colored by s which is incident to v_{i_0}. This implies that for any s, $1 < s < t$, there is a vertex or an edge with color s.

Case 2: there is a vertex v and there is an edge e' such that $\alpha(v) = 1$, $\alpha(e') = W_\tau(G)$.

Let $e' = v'w$ and P_2 be a simple path joining v with v', where

$$P_2 = (v_0, e_1, v_1, \ldots, v_{i-1}, e_i, v_i, \ldots, v_{k-1}, e_k, v_k), \ v_0 = v, \ v_k = v'.$$

If $\alpha(v_i) \neq s$, $i = 1, 2, \ldots, k$, and $\alpha(e_j) \neq s$, $j = 1, 2, \ldots, k$, then there exists an index i_1, $1 \leq i_1 < k$, such that $\alpha(e_{i_1}) < s$ and $\alpha(e_{i_1+1}) > s$. Hence, there is an edge of G colored by s which is incident to v_{i_1}. This implies that for any s, $1 < s < t$, there is a vertex or an edge with color s.

Case 3: there is an edge e and there is a vertex v' such that $\alpha(e) = 1$, $\alpha(v') = W_\tau(G)$.

Let $e = uv$ and P_3 be a simple path joining v with v', where

$$P_3 = (v_0, e_1, v_1, \ldots, v_{i-1}, e_i, v_i, \ldots, v_{k-1}, e_k, v_k), \ v_0 = v, \ v_k = v'.$$

If $\alpha(v_i) \neq s$, $i = 0, 1, \ldots, k-1$, and $\alpha(e_j) \neq s$, $j = 1, 2, \ldots, k$, then there exists an index i_2, $1 \leq i_2 < k$, such that $\alpha(e_{i_2}) < s$ and $\alpha(e_{i_2+1}) > s$. Hence, there is an edge of G colored by s which is incident to v_{i_2}. This implies that for any s, $1 < s < t$, there is a vertex or an edge with color s.

Case 4: there are edges $e, e' \in E(G)$ such that $\alpha(e) = 1$, $\alpha(e') = W_\tau(G)$.

Let $e = uv$, $e = v'w$. Without loss of generality we may assume that a simple path P_4 joining e with e' joins v with v', where

$$P_4 = (v_0, e_1, v_1, \ldots, v_{i-1}, e_i, v_i, \ldots, v_{k-1}, e_k, v_k), \ v_0 = v, \ v_k = v'.$$

If $\alpha(v_i) \neq s$, $i = 0, 1, \ldots, k$, and $\alpha(e_j) \neq s$, $j = 1, 2, \ldots, k$, then there exists an index i_3, $1 \leq i_3 < k$, such that $\alpha(e_{i_3}) < s$ and $\alpha(e_{i_3+1}) > s$. Hence, there is an edge of G colored by s which is incident to v_{i_3}. This implies that for any s, $1 < s < t$, there is a vertex or an edge with color s. $\square$

Now we show that there is an intimate connection between interval total colorings of graphs and interval edge-colorings of certain bipartite graphs.

Let G be a simple graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Define an auxiliary graph H as follows:

$$V(H) = U \cup W, \text{ where}$$

$$U = \{u_1, u_2, \ldots, u_n\}, \ W = \{w_1, w_2, \ldots, w_n\} \text{ and}$$

$$E(H) = \{u_i w_j, u_j w_i |\ v_i v_j \in E(G), 1 \le i \le n, 1 \le j \le n\} \cup \{u_i w_i |\ 1 \le i \le n\}.$$

Clearly, H is a bipartite graph with $|V(H)| = 2|V(G)|$.

Theorem 5 *If α is an interval total t-coloring of the graph G, then there is an interval t-coloring β of the bipartite graph H.*

Proof. For the proof, we define an edge-coloring β of the graph H as follows:

(1) $\beta(u_i w_j) = \beta(u_j w_i) = \alpha(v_i v_j)$ for every edge $v_i v_j \in E(G)$,

(2) $\beta(u_i w_i) = \alpha(v_i)$ for $i = 1, 2, \ldots, n$.

It is easy to see that β is an interval t-coloring of the graph H. $\square$

This theorem shows that any interval total t-coloring of a graph G can be transform into an interval t-coloring of the bipartite graph H.

Corollary 6 *If G is a connected graph and $G \in \mathfrak{T}$, then*

$$W_\tau(G) \le 2|V(G)| - 1.$$

Proof. Let α be an interval total $W_\tau(G)$-coloring of the graph G. By Theorem 5, β is an interval $W_\tau(G)$-coloring of the graph H. Since H is a connected bipartite graph with $|V(H)| = 2|V(G)|$ and $H \in \mathfrak{N}$, by Theorem 1, we have

$$W_\tau(G) \le |V(H)| - 1 = 2|V(G)| - 1, \text{ thus}$$

$$W_\tau(G) \le 2|V(G)| - 1.$$

$\square$

Remark 7 *Note that the upper bound in Corollary 6 is sharp for simple paths P_n, since $W_\tau(P_n) = 2n - 1$ for any $n \in \mathbf{N}$.*

Corollary 8 *If G is a connected r-regular graph with $|V(G)| \ge 2r + 2$ and $G \in \mathfrak{T}$, then*

$$W_\tau(G) \le 2|V(G)| - 3.$$

Proof. Let α be an interval total $W_\tau(G)$-coloring of the graph G. By Theorem 5, β is an interval $W_\tau(G)$-coloring of the graph H. Since H is a connected $(r + 1)$-regular bipartite graph with $|V(H)| \ge 2(2r + 2)$ and $H \in \mathfrak{N}$, by Theorem 2, we have

$$W_\tau(G) \le |V(H)| - 3 = 2|V(G)| - 3, \text{ thus}$$

$$W_\tau(G) \le 2|V(G)| - 3.$$

$\square$

Next we derive some upper bounds for $W_\tau(G)$ depending on degrees and diameter of a connected graph G.

Theorem 9 *If G is a connected graph and $G \in \mathfrak{T}$, then*

$$W_\tau(G) \leq 1 + \max_{P \in \mathbf{P}} \sum_{v \in V(P)} d_G(v),$$

where $\mathbf{P}$ is the set of all shortest paths in the graph G.

Proof. Consider an interval total $W_\tau(G)$-coloring α of G. We distinguish four possible cases.

Case 1: there are vertices $v, v' \in V(G)$ such that $\alpha(v) = 1$, $\alpha(v') = W_\tau(G)$.
Let P_1 be a shortest path joining v with v', where

$$P_1 = (v_1, e_1, v_2, e_2, \ldots, v_i, e_i, v_{i+1}, \ldots, v_k, e_k, v_{k+1}), \ v_1 = v, \ v_{k+1} = v'.$$

Note that

$$\alpha(e_1) \leq 1 + d_G(v_1),$$
$$\alpha(e_2) \leq \alpha(e_1) + d_G(v_2),$$
$$\cdots\cdots\cdots\cdots$$
$$\alpha(e_i) \leq \alpha(e_{i-1}) + d_G(v_i),$$
$$\cdots\cdots\cdots\cdots$$
$$\alpha(e_k) \leq \alpha(e_{k-1}) + d_G(v_k),$$
$$W_\tau(G) = \alpha(v') = \alpha(v_{k+1}) \leq \alpha(e_k) + d_G(v_{k+1}).$$

By summing these inequalities, we obtain

$$W_\tau(G) \leq 1 + \sum_{i=1}^{k+1} d_G(v_i) \leq 1 + \max_{P \in \mathbf{P}} \sum_{v \in V(P)} d_G(v).$$

Case 2: there is a vertex v and there is an edge e' such that $\alpha(v) = 1$, $\alpha(e') = W_\tau(G)$.
Let $e' = v'w$ and P_2 be a shortest path joining v with v', where

$$P_2 = (v_1, e_1, v_2, e_2, \ldots, v_i, e_i, v_{i+1}, \ldots, v_k, e_k, v_{k+1}), \ v_1 = v, \ v_{k+1} = v'.$$

Note that

$$\alpha(e_1) \leq 1 + d_G(v_1),$$
$$\alpha(e_2) \leq \alpha(e_1) + d_G(v_2),$$
$$\cdots\cdots\cdots\cdots$$
$$\alpha(e_i) \leq \alpha(e_{i-1}) + d_G(v_i),$$
$$\cdots\cdots\cdots\cdots$$
$$\alpha(e_k) \leq \alpha(e_{k-1}) + d_G(v_k),$$
$$W_\tau(G) = \alpha(e') = \alpha(v_{k+1}w) \leq \alpha(e_k) + d_G(v_{k+1}).$$

By summing these inequalities, we obtain

$$W_\tau(G) \leq 1 + \sum_{i=1}^{k+1} d_G(v_i) \leq 1 + \max_{P \in \mathbf{P}} \sum_{v \in V(P)} d_G(v).$$

Case 3: there is an edge e and there is a vertex v' such that $\alpha(e) = 1$, $\alpha(v') = W_\tau(G)$.
Let $e = uv$ and P_3 be a shortest path joining v with v', where

$$P_3 = (v_1, e_1, v_2, e_2, \ldots, v_i, e_i, v_{i+1}, \ldots, v_k, e_k, v_{k+1}), \ v_1 = v, \ v_{k+1} = v'.$$

Note that

$$\alpha(e_1) \leq 1 + d_G(v_1),$$
$$\alpha(e_2) \leq \alpha(e_1) + d_G(v_2),$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\alpha(e_i) \leq \alpha(e_{i-1}) + d_G(v_i),$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\alpha(e_k) \leq \alpha(e_{k-1}) + d_G(v_k),$$
$$W_\tau(G) = \alpha(v') = \alpha(v_{k+1}) \leq \alpha(e_k) + d_G(v_{k+1}).$$

By summing these inequalities, we obtain

$$W_\tau(G) \leq 1 + \sum_{i=1}^{k+1} d_G(v_i) \leq 1 + \max_{P \in \mathbf{P}} \sum_{v \in V(P)} d_G(v).$$

Case 4: there are edges $e, e' \in E(G)$ such that $\alpha(e) = 1$, $\alpha(e') = W_\tau(G)$.
Let $e = uv$ and $e' = v'w$. Without loss of generality we may assume that a shortest path P_4 joining e and e' joins v and v', where

$$P_4 = (v_1, e_1, v_2, e_2, \ldots, v_i, e_i, v_{i+1}, \ldots, v_k, e_k, v_{k+1}), \ v_1 = v, \ v_{k+1} = v'.$$

Note that

$$\alpha(e_1) \leq 1 + d_G(v_1),$$
$$\alpha(e_2) \leq \alpha(e_1) + d_G(v_2),$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\alpha(e_i) \leq \alpha(e_{i-1}) + d_G(v_i),$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\alpha(e_k) \leq \alpha(e_{k-1}) + d_G(v_k),$$
$$W_\tau(G) = \alpha(e') = \alpha(v'w) \leq \alpha(e_k) + d_G(v_{k+1}).$$

By summing these inequalities, we obtain

$$W_\tau(G) \leq 1 + \sum_{i=1}^{k+1} d_G(v_i) \leq 1 + \max_{P \in \mathbf{P}} \sum_{v \in V(P)} d_G(v).$$

$\square$

Corollary 10 *If G is a connected graph and $G \in \mathfrak{T}$, then $W_\tau(G) \leq 1 + (diam(G) + 1)\Delta(G)$.*

Now we give an upper bound on $W_\tau(G)$ for graphs with a unique universal vertex.

Theorem 11 *If G is a graph with a unique universal vertex u and $G \in \mathfrak{T}$, then $W_\tau(G) \leq |V(G)| + 2k(G)$, where $k(G) = \max_{v \in V(G)(v \neq u)} d_G(v)$.*

Proof. Let α be an interval total $W_\tau(G)$-coloring of the graph G.

Consider the vertex u. We show that $1 \leq \min S[u, \alpha] \leq k(G) + 1$.

Suppose, to the contrary, that $\min S[u, \alpha] \geq k(G) + 2$. Since $d_G(v) \leq k(G)$ for any $v \in V(G)(v \neq u)$, then $\min S[v, \alpha] \geq 2$ for any $v \in V(G)(v \neq u)$, which is a contradiction.

Now, we have

$$1 \leq \min S[u, \alpha] \leq k(G) + 1,$$

hence,

$$|V(G)| \leq \max S[u, \alpha] \leq |V(G)| + k(G).$$

This implies that $\max S[v, \alpha] \leq |V(G)| + 2k(G)$ for any $v \in V(G)(v \neq u)$. $\square$

In the next theorem we prove that regular bipartite graphs, trees and complete bipartite graphs are interval total colorable.

Theorem 12 *The set $\mathfrak{T}$ contains all regular bipartite graphs, trees and complete bipartite graphs.*

Proof. First we prove that if G is an r-regular bipartite graph with bipartition (U, V), then G has an interval total $(r + 2)$-coloring.

Since G is an r-regular bipartite graph, we have $\chi'(G) = \Delta(G) = r$. Let α be a proper edge-coloring of G with colors $2, 3, \ldots, r + 1$. Clearly, $S(w, \alpha) = [2, r + 1]$ for each $w \in V(G)$.

Define a total coloring β of the graph G as follows:

1. for any $u \in U$, let $\beta(u) = 1$;
2. for any $e \in E(G)$, let $\beta(e) = \alpha(e)$;
3. for any $v \in V$, let $\beta(v) = r + 2$.

It is easy to see that β is an interval total $(r + 2)$-coloring of G.

Next we consider trees. Clearly, K_1 is a tree and has an interval total 1-coloring. Assume that T is a tree and $T \neq K_1$. Now we prove that T has an interval total $(\Delta(T) + 2)$-coloring.

We use induction on $|E(T)|$. Clearly, the statement is true for the case $|E(T)| = 1$. Suppose that $|E(T)| = k > 1$ and the statement is true for all trees T' with $|E(T')| < k$.

Suppose $e = uv \in E(T)$ and $d_T(u) = 1$. Let $T' = T - u$. Since $|E(T)| > 1$, we have $d_T(v) \geq 2$. Clearly, $d_{T'}(v) = d_T(v) - 1, \Delta(T') \leq \Delta(T)$ and $|E(T')| = |E(T)| - 1 < k$. Let α be an interval total $(\Delta(T') + 2)$-coloring of the tree T' (by induction hypothesis). Consider the vertex v. Let

$$S[v, \alpha] = \{s(1), s(2), \ldots, s(d_{T'}(v) + 1)\},$$

where $1 \leq s(1) < s(2) < \ldots < s(d_{T'}(v) + 1) \leq \Delta(T) + 2$. We consider three cases.

Case 1: $s(1) = 1$.

Clearly, $s(d_{T'}(v) + 1) = d_{T'}(v) + 1 = d_T(v)$. In this case we color the edge e with color $d_T(v) + 1$ and the vertex u with color $d_T(v) + 2$. It is easy to see that the obtained coloring is an interval total $(\Delta(T) + 2)$-coloring of the tree T.

Case 2: $s(1) = 2$.

Subcase 2.1: $\alpha(v) = 2$.

Clearly, $s(d_{T'}(v) + 1) = d_T(v) + 1$. In this case we color the edge e with color $d_T(v) + 2$ and the vertex u with color $d_T(v) + 1$. It is easy to see that the obtained coloring is an interval total $(\Delta(T) + 2)$-coloring of the tree T.

Subcase 2.2: $\alpha(v) \neq 2$ and $\Delta(T') = \Delta(T)$.

We color the edge e with color 1 and the vertex u with color 2. It is easy to see that obtained coloring is an interval total $(\Delta(T) + 2)$-coloring of the tree T.

Subcase 2.3: $\alpha(v) \neq 2$ and $\Delta(T') < \Delta(T)$.

We define a total coloring β of the tree T' in the following way:

1. $\forall w \in V(T')$ $\beta(w) = \alpha(w) + 1$;
2. $\forall e' \in E(T')$ $\beta(e') = \alpha(e') + 1$.

Now we color the edge e with color 2 and the vertex u with color 1. It is not difficult to see that the obtained coloring is an interval total $(\Delta(T) + 2)$-coloring of the tree T.

Case 3: $s(1) \geq 3$.

We color the edge e with color $s(1) - 1$ and the vertex u with color $s(1) - 2$. It is easy to see that the obtained coloring is an interval total $(\Delta(T) + 2)$-coloring of the tree T.

Finally, we prove that if $K_{m,n}$ is a complete bipartite graph, then it has an interval total $(m + n + 1)$-coloring.

Let $V(K_{m,n}) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$ and $E(K_{m,n}) = \{u_i v_j |\ 1 \leq i \leq m, 1 \leq j \leq n\}$.

Define a total coloring γ of the graph $K_{m,n}$ as follows:

1. for $i = 1, 2, \ldots, m$, let $\gamma(u_i) = i$;
2. for $j = 1, 2, \ldots, n$, let $\gamma(v_j) = m + 1 + j$;
3. for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, let $\gamma(u_i v_j) = i + j$.

It is easy to see that γ is an interval total $(m + n + 1)$-coloring of $K_{m,n}$. $\square$

Corollary 13 *If G is an r-regular bipartite graph, then $w_\tau(G) \leq r + 2$.*

Corollary 14 *If T is a tree, then $w_\tau(T) \leq \Delta(T) + 2$.*

Corollary 15 *$W_\tau(K_{m,n}) \geq m + n + 1$ for any $m, n \in \mathbf{N}$.*

From Corollary 13, we have that $w_\tau(G) \leq r + 2$ for any r-regular bipartite graph G. On the other hand, clearly, $w_\tau(G) \geq r + 1$. In [14, 19] it was proved that the problem of determining whether $\chi''(G) = r + 1$ is NP-complete even for cubic bipartite graphs. Thus, we can conclude that the verification whether $w_\tau(G) = r + 1$ for any r-regular $(r \geq 3)$ bipartite graph G is also NP-complete.

4. Exact values of $w_\tau(G)$ and $W_\tau(G)$

In this section we determine the exact values of $w_\tau(G)$ and $W_\tau(G)$ for simple cycles, complete graphs and wheels.

In [25] it was proved the following result.

Theorem 16 *For the simple cycle C_n,*

$$\chi''(C_n) = \begin{cases} 3, & \text{if } n = 3k, \\ 4, & \text{if } n \neq 3k. \end{cases}$$

Theorem 17 *For any $n \geq 3$, we have*

(1) $C_n \in \mathfrak{T}$,

(2) $w_\tau(C_n) = \begin{cases} 3, & \text{if } n = 3k, \\ 4, & \text{if } n \neq 3k, \end{cases}$

(3) $W_\tau(C_n) = n + 2$.

Proof. First we prove that C_n has either an interval total 3-coloring or an interval total 4-coloring.

Let $V(C_n) = \{v_1, v_2 \ldots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_1 v_n\}$. We consider three cases.

Case 1: $n = 3k$ $(k \in \mathbf{N})$.

Define a total coloring α of the graph C_n as follows:

$$\alpha(v_i) = \begin{cases} 2, & \text{if } i \equiv 0 \pmod 3, \\ 1, & \text{if } i \equiv 1 \pmod 3, \\ 3, & \text{if } i \equiv 2 \pmod 3, \end{cases}$$

for $i = 1, 2, \ldots, n$,

$$\alpha(v_j v_{j+1}) = \begin{cases} 3, & \text{if } j \equiv 0 \pmod 3, \\ 2, & \text{if } j \equiv 1 \pmod 3, \\ 1, & \text{if } j \equiv 2 \pmod 3, \end{cases}$$

for $j = 1, 2, \ldots, n - 1$, and $\alpha(v_1 v_n) = 3$.

Case 2: $n \neq 3k$ $(k \in \mathbf{N})$ and n is even.

Define a total coloring α of the graph C_n as follows:

$$\alpha(v_i) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod 2, \\ 1, & \text{if } i \equiv 1 \pmod 2, \end{cases}$$

for $i = 1, 2, \ldots, n$,

$$\alpha(v_j v_{j+1}) = \begin{cases} 2, & \text{if } j \equiv 0 \pmod 2, \\ 3, & \text{if } j \equiv 1 \pmod 2, \end{cases}$$

for $j = 1, 2, \ldots, n - 1$, and $\alpha(v_1 v_n) = 2$.

Case 3: $n \neq 3k$ ($k \in \mathbf{N}$) and n is odd.

Define a total coloring α of the graph C_n as follows:

$$\alpha(v_i) = \begin{cases} 4, & \text{if } i \equiv 0 \pmod{2}, \ i \neq n - 1, \\ 1, & \text{if } i \equiv 1 \pmod{2}, \ i \neq n, \\ 2, & \text{if } i = n - 1, \\ 3, & \text{if } i = n, \end{cases}$$

for $i = 1, 2, \ldots, n$,

$$\alpha(v_j v_{j+1}) = \begin{cases} 2, & \text{if } j \equiv 0 \pmod{2}, \ j \neq n - 1, \\ 3, & \text{if } j \equiv 1 \pmod{2}, \\ 4, & \text{if } j = n - 1, \end{cases}$$

for $j = 1, 2, \ldots, n - 1$, and $\alpha(v_1 v_n) = 2$.

It is easy to check that α is an interval total 3-coloring of the graph C_n, when $n = 3k$, and an interval total 4-coloring of the graph C_n, when $n \neq 3k$. Hence, for any $n \geq 3$, $C_n \in \mathfrak{T}$ and $w_\tau(C_n) \leq 3$ if $n = 3k$ and $w_\tau(C_n) \leq 4$ if $n \neq 3k$. On the other hand, by Theorem 16, and taking into account that $w_\tau(C_n) \geq \chi''(C_n)$, we have $w_\tau(C_n) \geq 3$ if $n = 3k$ and $w_\tau(C_n) \geq 4$ if $n \neq 3k$. Thus, (1) and (2) hold.

Let us prove (3).

Now we show that $W_\tau(C_n) \geq n + 2$ for any $n \geq 3$. For that, we consider two cases.

Case 1: n is even.

Define a total coloring β of the graph C_n as follows:

1. for $i = 1, 2, \ldots, \frac{n}{2}$, let

$$\beta(v_i) = 2i - 1, \ \beta(v_i v_{i+1}) = 2i$$

2. for $j = \frac{n}{2} + 1, \ldots, n$, let

$$\beta(v_j) = 2(n - j) + 4,$$

3. for $k = \frac{n}{2} + 1, \ldots, n - 1$, let

$$\beta(v_k v_{k+1}) = 2(n - k) + 3,$$

and $\beta(v_1 v_n) = 3$.

Case 2: n is odd.

Define a total coloring β of the graph C_n as follows:

1. for $i = 1, 2, \ldots, \lceil \frac{n}{2} \rceil + 1$, let

$$\beta(v_i) = 2i - 1,$$

2. for $j = \lceil \frac{n}{2} \rceil + 2, \ldots, n$, let

$$\beta(v_j) = 2(n - j) + 4,$$

3. for $k = 1, 2, \ldots, \lceil \frac{n}{2} \rceil$, let

$$\beta(v_k v_{k+1}) = 2k,$$

4. for $l = \lceil \frac{n}{2} \rceil + 1, \ldots, n - 1$, let

$$\beta(v_l v_{l+1}) = 2(n - l) + 3,$$

and $\beta(v_1 v_n) = 3$.

It is not difficult to see that β is an interval total $(n + 2)$-coloring of the graph C_n. Thus, $W_\tau(C_n) \geq n + 2$ for any $n \geq 3$. On the other hand, using Corollary 10, and taking into account that $diam(C_n) = \lfloor \frac{n}{2} \rfloor$ and $\Delta(C_n) = 2$, it is easy to show that $W_\tau(C_n) \leq n + 2$ for any $n \geq 3$. $\square$

In [5] it was proved the following result.

Theorem 18 *For the complete graph K_n,*

$$\chi''(K_n) = \begin{cases} n, & \text{if } n \text{ is odd,} \\ n + 1, & \text{if } n \text{ is even.} \end{cases}$$

Theorem 19 *For any $n \in \mathbf{N}$, we have*

(1) $K_n \in \mathfrak{T}$,

(2) $w_\tau(K_n) = \begin{cases} n, & \text{if } n \text{ is odd,} \\ \frac{3}{2}n, & \text{if } n \text{ is even,} \end{cases}$

(3) $W_\tau(K_n) = 2n - 1$.

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$.

First we show that K_n has an interval total $(2n - 1)$-coloring for any $n \in \mathbf{N}$. For that, we define a total coloring α of the graph K_n as follows:

1. for $i = 1, 2, \ldots, n$, let $\alpha(v_i) = 2i - 1$;
2. for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$, where $i \neq j$, let $\alpha(v_i v_j) = i + j - 1$.

It is easy to see that α is an interval total $(2n - 1)$-coloring of the graph K_n. This proves that $K_n \in \mathfrak{T}$ and $W_\tau(K_n) \geq 2n - 1$ for any $n \in \mathbf{N}$. On the other hand, using Corollary 10, and taking into account that $diam(K_n) = 1$ and $\Delta(K_n) = n - 1$, it is simple to show that $W_\tau(K_n) \leq 2n - 1$ for any $n \in \mathbf{N}$. Thus, (1) and (3) hold.

Let us prove (2). We consider two cases.

Case 1: n is odd.

Since K_n is a regular graph, by Theorem 18, we have $w_\tau(K_n) = \chi''(K_n) = n$.

Case 2: n is even.

Now we show that $w_\tau(K_n) \leq \frac{3}{2}n$.

Define a total coloring β of the graph K_n as follows:

1. for $i = 1, 2, \ldots, \frac{n}{2}$, let

$$\beta(v_i) = i;$$

2. for $j = \frac{n}{2} + 1, \ldots, n$, let

$$\beta(v_j) = \tfrac{n}{2} + j;$$

3. for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, $i < j$, $i + j$ is odd, and $i + j - 1 \le n$, let

$$\beta(v_i v_j) = \tfrac{n}{2} + \tfrac{i+j-1}{2};$$

4. for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, $i < j$, $i + j$ is odd, and $i + j - 1 > n$, let

$$\beta(v_i v_j) = \tfrac{i+j-1}{2};$$

5. for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, $i < j$, $i + j$ is even, and $i + j \le n$, let

$$\beta(v_i v_j) = \tfrac{i+j}{2};$$

6. for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, $i < j$, $i + j$ is even, and $i + j > n$, let

$$\beta(v_i v_j) = \tfrac{n}{2} + \tfrac{i+j}{2}.$$

Let us show that β is an interval total $\frac{3}{2}n$-coloring of the graph K_n.
Let $v_i \in V(K_n)$, where $1 \le i \le n$.
If i is even, by the definition of β, we have

$$
\begin{aligned}
S\left[v_i, \beta\right] \;=\; & \left(\bigcup_{1 \le l \le \frac{n+2-i}{2}} \left\{ \frac{n}{2} + \frac{i + (2l-1) - 1}{2} \right\} \right) \cup \left(\bigcup_{\frac{n+2-i}{2} < l \le \frac{n}{2}} \left\{ \frac{i + (2l-1) - 1}{2} \right\} \right) \cup \\
& \left(\bigcup_{1 \le l \le \frac{n-i}{2}, l \ne \frac{i}{2}} \left\{ \frac{i + 2l}{2} \right\} \right) \cup \left(\bigcup_{\frac{n-i}{2} < l \le \frac{n}{2}, l \ne \frac{i}{2}} \left\{ \frac{n}{2} + \frac{i + 2l}{2} \right\} \right) \cup \left\{ i + \frac{n}{2} sg\left(i - \frac{n}{2} \right) \right\} \\
=\; & \left[\frac{n+i}{2}, n \right] \cup \left[\frac{n}{2} + 1, \frac{n+i}{2} - 1 \right] \cup \left(\left[\frac{i}{2} + 1, \frac{n}{2} \right] \setminus \{i\} \right) \cup \\
& \left(\left[n + 1, \frac{i}{2} + n \right] \setminus \left\{ \frac{n}{2} + i \right\} \right) \cup \left\{ i + \frac{n}{2} sg\left(i - \frac{n}{2} \right) \right\} \\
=\; & \left[\frac{i}{2} + 1, \frac{i}{2} + n \right],
\end{aligned}
$$

and if i is odd, by the definition of β, we have

$$
\begin{aligned}
S\left[v_i, \beta\right] \;=\;& \left(\bigcup_{1 \leq l \leq \frac{n+1-i}{2}}\left\{\frac{n}{2}+\frac{i+2l-1}{2}\right\}\right) \cup \left(\bigcup_{\frac{n+1-i}{2}<l \leq \frac{n}{2}}\left\{\frac{i+2l-1}{2}\right\}\right) \cup \\
& \left(\bigcup_{1 \leq l \leq \frac{n+1-i}{2}, l \neq \frac{i+1}{2}}\left\{\frac{i+2l-1}{2}\right\}\right) \cup \left(\bigcup_{\frac{n+1-i}{2}<l \leq \frac{n}{2}, l \neq \frac{i+1}{2}}\left\{\frac{n}{2}+\frac{i+2l-1}{2}\right\}\right) \cup \\
& \left\{i+\frac{n}{2} sg\left(i-\frac{n}{2}\right)\right\} \\
=\;& \left[\frac{n+i+1}{2}, n\right] \cup \left[\frac{n}{2}+1, \frac{n+i-1}{2}\right] \cup \left(\left[\frac{i+1}{2}, \frac{n}{2}\right] \setminus\{i\}\right) \cup \\
& \left(\left[n+1, \frac{i-1}{2}+n\right] \setminus\left\{\frac{n}{2}+i\right\}\right) \cup \left\{i+\frac{n}{2} sg\left(i-\frac{n}{2}\right)\right\} \\
=\;& \left[\frac{i+1}{2}, \frac{i-1}{2}+n\right].
\end{aligned}
$$

This shows that β is an interval total $\frac{3}{2}n$-coloring of the graph K_n.

Next we prove that $w_\tau(K_n) \geq \frac{3}{2}n$.

Suppose, to the contrary, that γ is an interval total $w_\tau(K_n)$-coloring of the graph K_n, where $n \leq w_\tau(K_n) \leq \frac{3}{2}n - 1$.

Since $w_\tau(K_n) \geq \chi''(K_n)$, by Theorem 18, we have $n + 1 \leq w_\tau(K_n) \leq \frac{3}{2}n - 1$.

Consider the vertices $v_1, v_2, \ldots, v_n$. It is clear that

$$1 \leq \min S[v_i, \gamma] \leq w_\tau(K_n) - n + 1 \text{ for } i = 1, 2, \ldots, n.$$

Hence, $\{w_\tau(K_n) - n + 1, \ldots, n\} \subseteq S[v_i, \gamma]$ for $i = 1, 2, \ldots, n$. Let us show that none of the vertices $v_1, v_2, \ldots, v_n$ is colored by j, $j = w_\tau(K_n) - n + 1, \ldots, n$. Suppose that $\gamma(v_{i_0}) = j_0$, $j_0 \in \{w_\tau(K_n) - n + 1, \ldots, n\}$. Clearly, $\gamma(v_i) \neq j_0$ for $i = 1, 2, \ldots, n$ and $i \neq i_0$. This implies that any vertex v_i, except v_{i_0}, is incident to an edge with color j_0, which is a contradiction. The contradiction shows that $\gamma(v_i) \notin \{w_\tau(K_n) - n + 1, \ldots, n\}$ for $i = 1, 2, \ldots, n$. Hence,

$$\gamma(v_i) \in \{1, 2, \ldots, w_\tau(K_n) - n\} \cup \{n + 1, \ldots, w_\tau(K_n)\} \text{ for } i = 1, 2, \ldots, n.$$

On the other hand, since $\chi(K_n) = n$, we have

$$|\{1, 2, \ldots, w_\tau(K_n) - n\}| + |\{n + 1, \ldots, w_\tau(K_n)\}| \geq n,$$

thus $w_\tau(K_n) \geq \frac{3}{2}n$, which is a contradiction. $\square$

Theorem 20 *For any $n \in \mathbf{N}$,*

(1) *if $2n - 1 \leq t \leq 4n - 3$, then $K_{2n-1} \in \mathfrak{T}_t$,*

(2) *if $3n \leq t \leq 4n - 1$, then $K_{2n} \in \mathfrak{T}_t$.*

Proof. First we prove (1). For that, we transform the interval total $(4n-3)$-coloring α of the graph K_{2n-1} constructed in the proof of Theorem 19, into an interval total t-coloring β of the same graph.

For every $v \in V(K_{2n-1})$, we set:

$$\beta(v) = \begin{cases} \alpha(v), & \text{if } 1 \le \alpha(v) \le t, \\ \alpha(v) - 2n + 1, & \text{if } t+1 \le \alpha(v) \le 4n - 3. \end{cases}$$

For every $e \in E(K_{2n-1})$, we set:

$$\beta(e) = \begin{cases} \alpha(e), & \text{if } 1 \le \alpha(e) \le t, \\ \alpha(e) - 2n + 1, & \text{if } t+1 \le \alpha(e) \le 4n - 3. \end{cases}$$

It is easy to see that β is an interval total t-coloring of the graph K_{2n-1}.

Let us prove (2).

For that, we transform the interval total $3n$-coloring β of the graph K_{2n} constructed in the proof of Theorem 19, into an interval total t-coloring γ of the same graph.

Define a total coloring γ of the graph K_{2n} as follows:

1. for $i = 1, 2, \ldots, 2n$, let

$$\gamma(v_i) = \begin{cases} \beta(v_i) + t - 3n, & \text{if } \beta(v_i) + t - 3n \le 2i - 1, \\ 2i - 1, & \text{if } \beta(v_i) + t - 3n > 2i - 1; \end{cases}$$

2. for $i = 1, 2, \ldots, 2n - 1$, $j = 1, 2, \ldots, 2n - 1$, $i \ne j$, and $i + j - 1 \le 2(t - 3n) + 1$, let

$$\gamma(v_i v_j) = i + j - 1;$$

3. for $i = 1, 2, \ldots, 2n$, $j = 1, 2, \ldots, 2n$, $i \ne j$, and $2(t - 3n) + 1 < i + j - 1 < 2n$, let

$$\gamma(v_i v_j) = \begin{cases} \beta(v_i v_j) + t - 3n, & \text{if } i + j \text{ is even}, \\ \beta(v_i v_j), & \text{if } i + j \text{ is odd}; \end{cases}$$

4. for $i = 1, 2, \ldots, 2n$, $j = 1, 2, \ldots, 2n$, $i \ne j$, and $2n \le i + j - 1 \le 2n + 2(t - 3n) + 1$, let

$$\gamma(v_i v_j) = i + j - 1;$$

5. for $i = 3, 4, \ldots, 2n$, $j = 3, 4, \ldots, 2n$, $i \ne j$, and $i + j - 1 > 2n + 2(t - 3n) + 1$, let

$$\gamma(v_i v_j) = \begin{cases} \beta(v_i v_j) + t - 3n, & \text{if } i + j \text{ is even}, \\ \beta(v_i v_j), & \text{if } i + j \text{ is odd}; \end{cases}$$

It can be easily verified that γ is an interval total t-coloring of the graph K_{2n}. $\square$

Finally, we obtain the exact values of $w_\tau(G)$ and $W_\tau(G)$ for wheels. Recall that a wheel W_n ($n \ge 4$) is defined as follows:

$$V(W_n) = \{u, v_1, v_2, \ldots, v_{n-1}\} \text{ and}$$
$$E(W_n) = \{uv_i | \ 1 \le i \le n - 1\} \cup \{v_i v_{i+1} | \ 1 \le i \le n - 2\} \cup \{v_1 v_{n-1}\}.$$

Lemma 21 *For any $n \geq 4$, we have $W_n \in \mathfrak{T}$ and*

$$w_\tau(W_n) = \begin{cases} n + 2, & \text{if } n = 4, \\ n, & \text{if } n \geq 5. \end{cases}$$

Proof. Clearly, $W_4 = K_4$, hence, by Theorem 19, we have $W_4 \in \mathfrak{T}$ and $w_\tau(W_4) = w_\tau(K_4) = 6$.

Assume that $n \geq 5$.

For the proof of the lemma we construct an interval total n-coloring of the graph W_n. We consider two cases.

Case 1: n is even.

Define a total coloring α of the graph W_n as follows:

1) $\alpha(u) = n$, $\alpha(v_1) = 2$ and for $i = 2, \ldots, \frac{n}{2} - 1$, let $\alpha(v_i) = 2i + 1$;

2) $\alpha(v_{\frac{n}{2}}) = n - 2$, $\alpha(v_{\frac{n}{2}+1}) = n - 4$, and for $j = \frac{n}{2} + 2, \ldots, n - 1$, let $\alpha(v_j) = 2(n - j + 1)$;

3) for $k = 1, 2, \ldots, \frac{n}{2}$, let

$$\alpha(uv_k) = 2k - 1;$$

4) for $l = \frac{n}{2} + 1, \ldots, n - 1$, let

$$\alpha(uv_l) = 2(n - l);$$

5) for $p = 1, \ldots, \frac{n}{2} - 1$, let

$$\alpha(v_p v_{p+1}) = 2(p + 1) \text{ and } \alpha\left(v_{\frac{n}{2}} v_{\frac{n}{2}+1}\right) = n - 3;$$

6) for $q = \frac{n}{2} + 1, \ldots, n - 2$, let

$$\alpha(v_q v_{q+1}) = 2(n - q) + 1 \text{ and } \alpha(v_1 v_{n-1}) = 3.$$

Case 2: n is odd.

Define a total coloring β of the graph W_n as follows:

1) $\beta(u) = n$, $\beta(v_1) = 2$ and for $i = 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, let $\beta(v_i) = 2i + 1$;

2) $\beta(v_{\lfloor \frac{n}{2} \rfloor}) = n - 4$, $\beta(v_{\lceil \frac{n}{2} \rceil}) = n - 2$ and for $j = \lceil \frac{n}{2} \rceil + 1, \ldots, n - 1$, let $\beta(v_j) = 2(n - j + 1)$;

3) for $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, let

$$\beta(uv_k) = 2k - 1;$$

4) for $l = \lceil \frac{n}{2} \rceil, \ldots, n - 1$, let

$$\beta(uv_l) = 2(n - l);$$

5) for $p = 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, let

$$\beta(v_p v_{p+1}) = 2(p + 1) \text{ and } \beta\left(v_{\lfloor \frac{n}{2} \rfloor} v_{\lceil \frac{n}{2} \rceil}\right) = n - 3;$$

6) for $q = \lceil \frac{n}{2} \rceil, \ldots, n - 2$, let

$$\beta(v_q v_{q+1}) = 2(n - q) + 1 \text{ and } \beta(v_1 v_{n-1}) = 3.$$

It is not difficult to check that α is an interval total n-coloring of the graph W_n, when n is even, and β is an interval total n-coloring of the graph W_n, when n is odd. Hence, $W_n \in \mathfrak{T}$. On the other hand, clearly, $w_\tau(W_n) \geq \chi''(W_n) = \Delta(W_n) + 1 = n$, thus $w_\tau(W_n) = n$. $\square$

Lemma 22 *For any $n \geq 5$, we have $W_n \in \mathfrak{T}_{n+1} \cap \mathfrak{T}_{n+2}$.*

Proof. First we show that $W_n \in \mathfrak{T}_{n+2}$ for any $n \geq 5$.
Define a total coloring α of the graph W_n as follows:
1) $\alpha(u) = 1$, $\alpha(v_1) = 3$, $\alpha(v_{\lceil \frac{n}{2} \rceil}) = n - 1$ and for $i = 2, \ldots, \lceil \frac{n}{2} \rceil - 1$, let $\alpha(v_i) = 2(i + 1)$;
2) for $j = \lceil \frac{n}{2} \rceil + 1, \ldots, n - 1$, let $\alpha(v_j) = 2(n - j) + 3$;
3) for $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, let

$$\alpha(uv_k) = 2k;$$

4) for $l = \lfloor \frac{n}{2} \rfloor + 1, \ldots, n - 1$, let

$$\alpha(uv_l) = 2(n - l) + 1;$$

5) for $p = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$, let

$$\alpha(v_p v_{p+1}) = 2p + 3;$$

6) for $q = \lfloor \frac{n-1}{2} \rfloor + 1, \ldots, n - 2$

$$\alpha(v_q v_{q+1}) = 2(n - q + 1) \text{ and } \alpha(v_1 v_{n-1}) = 4.$$

It is easily seen that α is an interval total $(n + 2)$-coloring of the graph W_n.
Now we show that $W_n \in \mathfrak{T}_{n+1}$ for any $n \geq 5$.
Define a total coloring β of the graph W_n as follows:
1) for $\forall v \in V(W_n)$, let $\beta(v) = \alpha(v)$;
2) for $\forall e \in E(W_n)$, let

$$\beta(e) = \begin{cases} \alpha(e), & \text{if } \alpha(e) \neq n + 2, \\ n - 2, & \text{otherwise.} \end{cases}$$

It is easily seen that β is an interval total $(n + 1)$-coloring of the graph W_n. $\square$

Lemma 23 *For any $n \geq 4$, we have $W_\tau(W_n) \geq n + 3$.*

Proof. Clearly, for the proof of the lemma it suffices to construct an interval total $(n+3)$-coloring of the graph W_n for $n \geq 4$. We consider two cases.
Case 1: n is even.
Define a total coloring α of the graph W_n as follows:
1) for $i = 1, 2, \ldots, \frac{n}{2} + 1$, let $\alpha(v_i) = 2i - 1$;
2) for $j = \frac{n}{2} + 2, \ldots, n - 1$, let $\alpha(v_j) = 2(n - j + 1)$;
3) for $k = 1, 2, \ldots, \frac{n}{2}$, let

$$\alpha(v_k v_{k+1}) = 2k;$$

4) for $l = \frac{n}{2} + 1, \ldots, n - 2$, let

$$\alpha\left(v_l v_{l+1}\right) = 2(n-l) + 1 \text{ and } \alpha\left(v_1 v_{n-1}\right) = 3;$$

5) for $p = 2, \ldots, \frac{n}{2}$, let

$$\alpha\left(uv_p\right) = 2p + 1 \text{ and } \alpha\left(uv_1\right) = 4;$$

6) for $q = \frac{n}{2} + 1, \ldots, n-1$, let

$$\alpha\left(uv_q\right) = 2(n-q+2) \text{ and } \alpha(u) = n+3.$$

Case 2: n is odd.

Define a total coloring β of the graph W_n as follows:

1) for $i = 1, 2, \ldots, \lfloor\frac{n}{2}\rfloor$, let $\beta\left(v_i\right) = 2i - 1$, $\beta\left(v_i v_{i+1}\right) = 2i$;

2) for $j = \lceil\frac{n}{2}\rceil, \ldots, n-1$, let $\beta(v_j) = 2(n-j+1)$;

3) for $k = \lceil\frac{n}{2}\rceil, \ldots, n-2$, let

$$\beta\left(v_k v_{k+1}\right) = 2(n-k) + 1 \text{ and } \beta\left(v_1 v_{n-1}\right) = 3;$$

4) for $p = 2, 3, \ldots, \lceil\frac{n}{2}\rceil$, let

$$\beta\left(uv_p\right) = 2p + 1 \text{ and } \beta\left(uv_1\right) = 4;$$

5) for $q = \lceil\frac{n}{2}\rceil + 1, \ldots, n-1$, let

$$\beta\left(uv_q\right) = 2(n-q+2) \text{ and } \beta(u) = n+3.$$

It is not difficult to check that α is an interval total $(n+3)$-coloring of the graph W_n, when n is even, and β is an interval total $(n+3)$-coloring of the graph W_n, when n is odd. $\square$

Remark 24 *Easy analysis shows that if $4 \leq n \leq 8$, then $W_\tau(W_n) = n+3$.*

Lemma 25 *For any $n \geq 9$, we have $W_\tau(W_n) \geq n+4$.*

Proof. Clearly, for the proof of the lemma it suffices to construct an interval total $(n+4)$-coloring of the graph W_n for $n \geq 9$. We consider two cases.

Case 1: n is even.

Define a total coloring α of the graph W_n as follows:

1) $\alpha(u) = 7$, $\alpha(v_1) = 1$, $\alpha(v_2) = 6$, $\alpha(v_3) = 8$ and for $i = 4, \ldots, \frac{n}{2} - 2$, let $\alpha\left(v_i\right) = 2i + 1$;

2) $\alpha(v_{\frac{n}{2}-1}) = n+2$, $\alpha(v_{\frac{n}{2}}) = n+4$ and for $j = \frac{n}{2} + 1, \ldots, n-2$, let $\alpha(v_j) = 2(n-j)$, $\alpha(v_{n-1}) = 3$;

3) $\alpha(uv_1) = 3$, $\alpha(uv_2) = 5$ and for $k = 3, \ldots, \frac{n}{2} - 1$, let

$$\alpha\left(uv_k\right) = 2k + 3;$$

4) for $l = \frac{n}{2}, \ldots, n-1$, let

$$\alpha\left(uv_l\right) = 2(n-l+1);$$

5) $\alpha(v_1 v_2) = 4$, $\alpha(v_2 v_3) = 7$ and for $p = 3, \ldots, \frac{n}{2} - 2$, let

$$\alpha\left(v_p v_{p+1}\right) = 2(p+2);$$

6) for $q = \frac{n}{2} - 1, \ldots, n - 2$, let

$$\alpha (v_q v_{q+1}) = 2(n - q) + 1 \text{ and } \alpha (v_1 v_{n-1}) = 2.$$

Case 2: n is odd.

Define a total coloring β of the graph W_n as follows:

1) $\beta(u) = 7$, $\beta(v_1) = 1$, $\beta(v_2) = 6$, $\beta(v_3) = 8$ and for $i = 4, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, let $\beta (v_i) = 2i + 1$;

2) $\beta(v_{\lfloor \frac{n}{2} \rfloor}) = n + 4$, $\beta(v_{\lceil \frac{n}{2} \rceil}) = n + 2$ and for $j = \lceil \frac{n}{2} \rceil + 1, \ldots, n - 2$, let $\beta(v_j) = 2(n - j)$, $\beta(v_{n-1}) = 3$;

3) $\beta(uv_1) = 3$, $\beta(uv_2) = 5$ and for $k = 3, \ldots, \lfloor \frac{n}{2} \rfloor$, let

$$\beta (uv_k) = 2k + 3;$$

4) for $l = \lceil \frac{n}{2} \rceil, \ldots, n - 1$, let

$$\beta (uv_l) = 2(n - l + 1);$$

5) $\beta(v_1 v_2) = 4$, $\beta(v_2 v_3) = 7$ and for $p = 3, \ldots, \lfloor \frac{n}{2} \rfloor$, let

$$\beta (v_p v_{p+1}) = 2(p + 2);$$

6) for $q = \lceil \frac{n}{2} \rceil, \ldots, n - 2$, let

$$\beta (v_q v_{q+1}) = 2(n - q) + 1 \text{ and } \beta (v_1 v_{n-1}) = 2.$$

It is easy to check that α is an interval total $(n + 4)$-coloring of the graph W_n, when n is even, and β is an interval total $(n + 4)$-coloring of the graph W_n, when n is odd. $\square$

Lemma 26 *For any $n \geq 4$, we have $W_\tau(W_n) \leq n + 4$.*

Proof. First, by Theorem 11, we have $W_\tau(W_n) \leq n + 6$ for any $n \geq 4$.

Next we prove that $W_n \notin \mathfrak{T}_{n+5}$.

Suppose, to the contrary, that α is an interval total $(n + 5)$-coloring of the graph W_n for $n \geq 4$.

Consider the vertex u. Clearly,

$$1 \leq \min S[u, \alpha] \leq 6,$$

hence

$$n \leq \max S[u, \alpha] \leq n + 5.$$

Proposition 3 implies that the following three cases are possible:

1) $S[u, \alpha] = [6, n + 5]$;
2) $S[u, \alpha] = [5, n + 4]$;
3) $S[u, \alpha] = [4, n + 3]$.

Case 1: $S[u, \alpha] = [6, n + 5]$ or $S[u, \alpha] = [5, n + 4]$.

Clearly, $\alpha(uv_i) \geq 5$ for $i = 1, \ldots, n - 1$. This implies that $\min S[v_i, \alpha] \geq 2$ for $i = 1, \ldots, n - 1$, which is a contradiction.
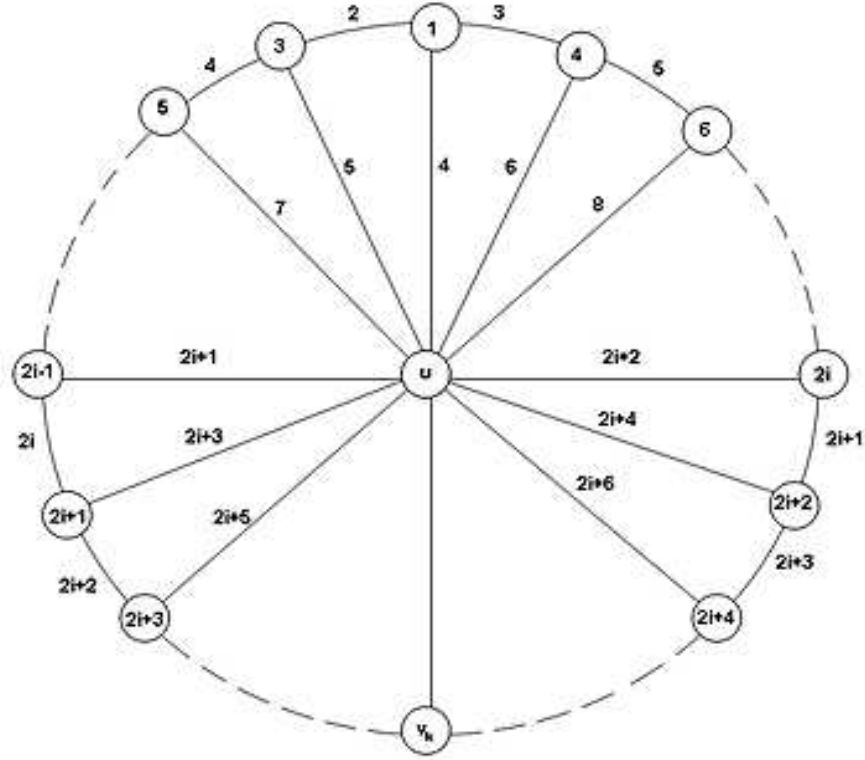
Figure 1.

Case 2: $S[u, \alpha] = [4, n + 3]$.

First we show that $\alpha(u) \neq 4$. Suppose that $\alpha(u) = 4$. This implies that $\alpha(uv_i) \geq 5$ for $i = 1, \ldots, n - 1$, which is a contradiction.

Let $e = uv_1$ and $\alpha(e) = 4$. Note that $\alpha(v_1) = 1$.

Without loss of generality, we may assume that $\alpha(v_1 v_2) = 2$, $\alpha(v_1 v_{n-1}) = 3$, $\alpha(uv_2) = 5$, $\alpha(uv_{n-1}) = 6$, and there is a vertex v_k such that either $\alpha(v_k) = n + 5$ or $\alpha(v_k v_{k+1}) = n + 5$ (see Fig. 1).

Let us consider simple paths

$$P_1 = (v_1, v_1 v_2, v_2, \ldots, v_k, v_k v_{k+1}, v_{k+1}) \text{ and}$$
$$P_2 = (v_{n-1}, v_{n-1} v_{n-2}, v_{n-2}, \ldots, v_{k+1}, v_{k+1} v_k, v_k),$$

where $1 \leq k \leq n - 2$.

Let us show that

1) $\alpha(v_i) = 2i - 1$, $\alpha(v_i v_{i+1}) = 2i$, $\alpha(uv_i) = 2i + 1$,

2) $\alpha(v_{n+1-i}) = 2i$, $\alpha(v_{n-i} v_{n+1-i}) = 2i + 1$, $\alpha(uv_{n+1-i}) = 2(i + 1)$,

for $i = 2, \ldots, k$.

We use induction on i. For $i = 2$, it suffices to prove that $\alpha(v_2) = 3$, $\alpha(v_2 v_3) = 4$, $\alpha(v_{n-1}) = 4$, $\alpha(v_{n-2} v_{n-1}) = 5$.

Consider the vertex v_2. Since $\alpha(v_1v_2) = 2$ and $\alpha(uv_2) = 5$, we have $\min S[v_2, \alpha] = 2$ and $\max S[v_2, \alpha] = 5$, hence $\{3, 4\} \subseteq S[v_2, \alpha]$. If we suppose that $\alpha(v_2) = 4$, then $\alpha(v_2v_3) = 3$ and $\max S[v_3, \alpha] < 7$, which contradicts $\max S[v_3, \alpha] \geq 7$. From this we have $\alpha(uv_3) = 7$ (see Fig. 1).

Now we consider the vertex v_{n-1}. Since $\alpha(v_1v_{n-1}) = 3$ and $\alpha(uv_{n-1}) = 6$, we have $\min S[v_{n-1}, \alpha] = 3$ and $\max S[v_{n-1}, \alpha] = 6$, hence $\{4, 5\} \subseteq S[v_{n-1}, \alpha]$. If we suppose that $\alpha(v_{n-1}) = 5$, then $\alpha(v_{n-2}v_{n-1}) = 4$ and $\max S[v_{n-2}, \alpha] < 8$, which contradicts $\max S[v_{n-2}, \alpha] \geq 8$ (see Fig. 1).

Suppose that the statements 1) and 2) are true for all i', $1 \leq i' \leq i$. We prove that the statements 1) and 2) are true for the case $i+1$, that is, $\alpha(v_{i+1}) = 2i+1$, $\alpha(v_{i+1}v_{i+2}) = 2i+2$, $\alpha(uv_{i+1}) = 2i + 3$ and $\alpha(v_{n-i}) = 2i + 2$, $\alpha(v_{n-i-1}v_{n-i}) = 2i + 3$, $\alpha(uv_{n-i}) = 2i + 4$. From the induction hypothesis we have:

1') $\alpha(v_j) = 2j - 1$, $\alpha(v_jv_{j+1}) = 2j$, $\alpha(uv_j) = 2j + 1$,
2') $\alpha(v_{n+1-j}) = 2j$, $\alpha(v_{n-j}v_{n+1-j}) = 2j + 1$, $\alpha(uv_{n+1-j}) = 2(j + 1)$,

for $j = 2, \ldots, i$.

1') and 2') implies that $\alpha(uv_{i+1}) = 2i + 3$ and $\alpha(uv_{n-i}) = 2i + 4$.

Consider the vertex v_{i+1}. Since $\alpha(v_iv_{i+1}) = 2i$ and $\alpha(uv_{i+1}) = 2i + 3$, we have $\min S[v_{i+1}, \alpha] = 2i$ and $\max S[v_{i+1}, \alpha] = 2i + 3$, hence $\{2i + 1, 2i + 2\} \subseteq S[v_{i+1}, \alpha]$. If we suppose that $\alpha(v_{i+1}) = 2i+2$, then $\alpha(v_{i+1}v_{i+2}) = 2i+1$ and $\max S[v_{i+2}, \alpha] < 2i+5$, which contradicts $\max S[v_{i+2}, \alpha] \geq 2i + 5$. From this we have $\alpha(uv_{i+2}) = 2i + 5$ (see Fig. 1).

Next we consider the vertex v_{n-i}. Since $\alpha(v_{n+1-i}v_{n-i}) = 2i+1$ and $\alpha(uv_{n-i}) = 2i+4$, we have $\min S[v_{n-i}, \alpha] = 2i+1$ and $\max S[v_{n-i}, \alpha] = 2i+4$, hence $\{2i+2, 2i+3\} \subseteq S[v_{n-i}, \alpha]$. If we suppose that $\alpha(v_{n-i}) = 2i + 3$, then $\alpha(v_{n-i-1}v_{n-i}) = 2i + 2$ and $\max S[v_{n-i-1}, \alpha] < 2i + 6$, which contradicts $\max S[v_{n-i-1}, \alpha] \geq 2i + 6$ (see Fig. 1).

By 1'), we have $k \geq \frac{n}{2} + 2$.

By 2'), we have $k \leq \frac{n}{2} - 1$.

It is easy to see that does not exist such an index k, which satisfy the aforementioned inequalities. This completes the prove of the case 2.

Similarly, it can be shown that $W_n \notin \mathfrak{T}_{n+6}$, hence $W_\tau(W_n) \leq n + 4$ for any $n \geq 4$. $\square$

From Lemmas 21-26 and Remark 24, we have the following result:

Theorem 27 *For $n \geq 4$, we have*

(1) $W_n \in \mathfrak{T}$,

(2) $w_\tau(W_n) = \begin{cases} n + 2, & \text{if } n = 4, \\ n, & \text{if } n \geq 5, \end{cases}$

(3) $W_\tau(W_n) = \begin{cases} n + 3, & \text{if } 4 \leq n \leq 8, \\ n + 4, & \text{if } n \geq 9, \end{cases}$

(4) *if $w_\tau(W_n) \leq t \leq W_\tau(W_n)$, then $W_n \in \mathfrak{T}_t$.*

REFERENCES

1. A.S. Asratian, R.R. Kamalian, Interval colorings of edges of a multigraph, Appl. Math. 5 (1987) 25-34 (in Russian).
2. A.S. Asratian, R.R. Kamalian, Investigation on interval edge-colorings of graphs, J. Combin. Theory Ser. B 62 (1994) 34-43.
3. A.S. Asratian, C.J. Casselgren, On interval edge colorings of (α, β)-biregular bipartite graphs, Discrete Mathematics 307 (2006) 1951-1956.
4. M. Behzad, Graphs and their chromatic numbers, Ph.D. thesis, Michigan State University, 1965.
5. M. Behzad, G. Chartrand, J.K. Cooper Jr., The colour numbers of complete graphs, J. London Math. Soc. 42 (1967) 226-228.
6. O.V. Borodin, On the total colouring of planar graphs, J. Reine Angew. Math. 394 (1989) 180-185.
7. O.V. Borodin, A.V. Kostochka, D.R. Woodall, Total colorings of planar graphs with large maximum degree, J. Graph Theory 26 (1997) 53-59.
8. K.H. Chew, H.P. Yap, Total chromatic number of complete r-partite graphs, J. Graph Theory 16 (1992) 629-634.
9. A.J.W. Hilton, H.R. Hind, The total chromatic number of graphs having large maximum degree, Discrete Mathematics 117 (1993) 127-140.
10. T.R. Jensen, B. Toft, Graph coloring problems, Wiley Interscience Series in Discrete Mathematics and Optimization, 1995.
11. A.V. Kostochka, The total coloring of a multigraph with maximal degree 4, Discrete Mathematics 17 (1977) 161-163.
12. A.V. Kostochka, The total chromatic number of any multigraph with maximum degree five is at most seven, Discrete Mathematics 162 (1996) 199-214.
13. L. Kowalik, J.-S. Sereni, R. Skrekovski, Total-colouring of plane graphs with maximum degree nine, SIAM J. Discrete Math. 22 (2008) 1462-1479.
14. C.J.H. McDiarmid, A. Sanchez-Arroyo, Total colouring regular bipartite graphs is NP-hard, Discrete Mathematics 124 (1994) 155-162.
15. M. Molloy, B. Reed, A bound on the total chromatic number, Combinatorica 18 (1998) 241-280.
16. P.A. Petrosyan, Interval total colorings of complete bipartite graphs, Proceedings of the CSIT Conference (2007) 84-85.
17. P.A. Petrosyan, Interval total colorings of certain graphs, Mathematical Problems of Computer Science 31 (2008) 122-129.
18. M. Rosenfeld, On the total coloring of certain graphs, Israel J. Math. 9 (1971) 396-402.
19. A. Sanchez-Arroyo, Determining the total colouring number is NP-hard, Discrete Mathematics 78 (1989) 315-319.
20. D.P. Sanders, Y. Zhao, On total 9-coloring planar graphs of maximum degree seven, J. Graph Theory 31 (1999) 67-73.
21. N. Vijayaditya, On the total chromatic number of a graph, J. London Math. Soc. (2) 3 (1971) 405-408.
22. V.G. Vizing, Chromatic index of multigraphs, Doctoral Thesis, Novosibirsk, 1965 (in Russian).

23. W. Wang, Total chromatic number of planar graphs with maximum degree ten, J. Graph Theory 54 (2006) 91-102.
24. D.B. West, Introduction to Graph Theory, Prentice-Hall, New Jersey, 1996.
25. H.P. Yap, Total colorings of graphs, Lecture Notes in Mathematics 1623, Springer-Verlag, Berlin, 1996.
26. Z. Zhang, J. Zhand, J. Wang, The total chromatic numbers of some graphs, Scientia Sinica A 31 (1988) 1434-1441.